

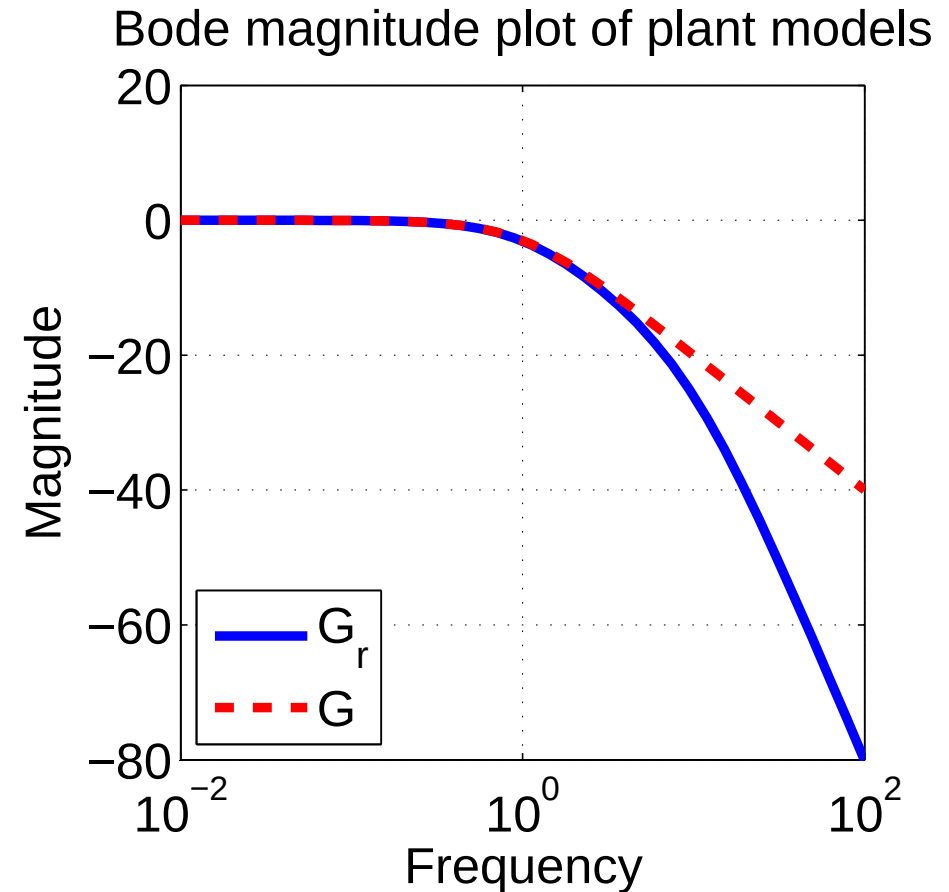
# Motivating Example

**Real system:**

$$G_{\text{real}}(s) = \frac{1}{s+1} \cdot \frac{1}{(0.1s+1)^2}$$

**Approximation:**

$$G(s) = \frac{1}{s+1}$$



Idea: Bode plots are similar, differences only for large freqs.

Place poles far from stability border in left half-plane to be on safe side

# Motivating Example

**Controller:**

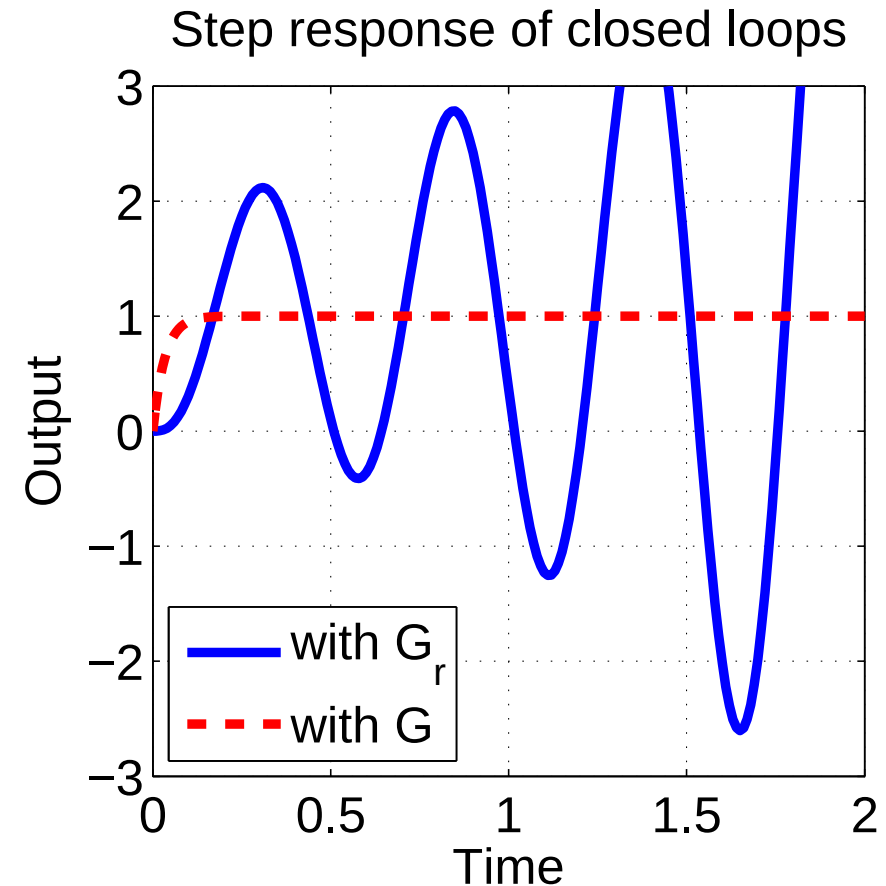
$$K(s) = \frac{30(s+1)}{s}$$

achieves

$$G_{cl}(s) = \frac{GK}{1+GK} = \frac{30}{s+30}$$

i.e. pole at  $s = -30$

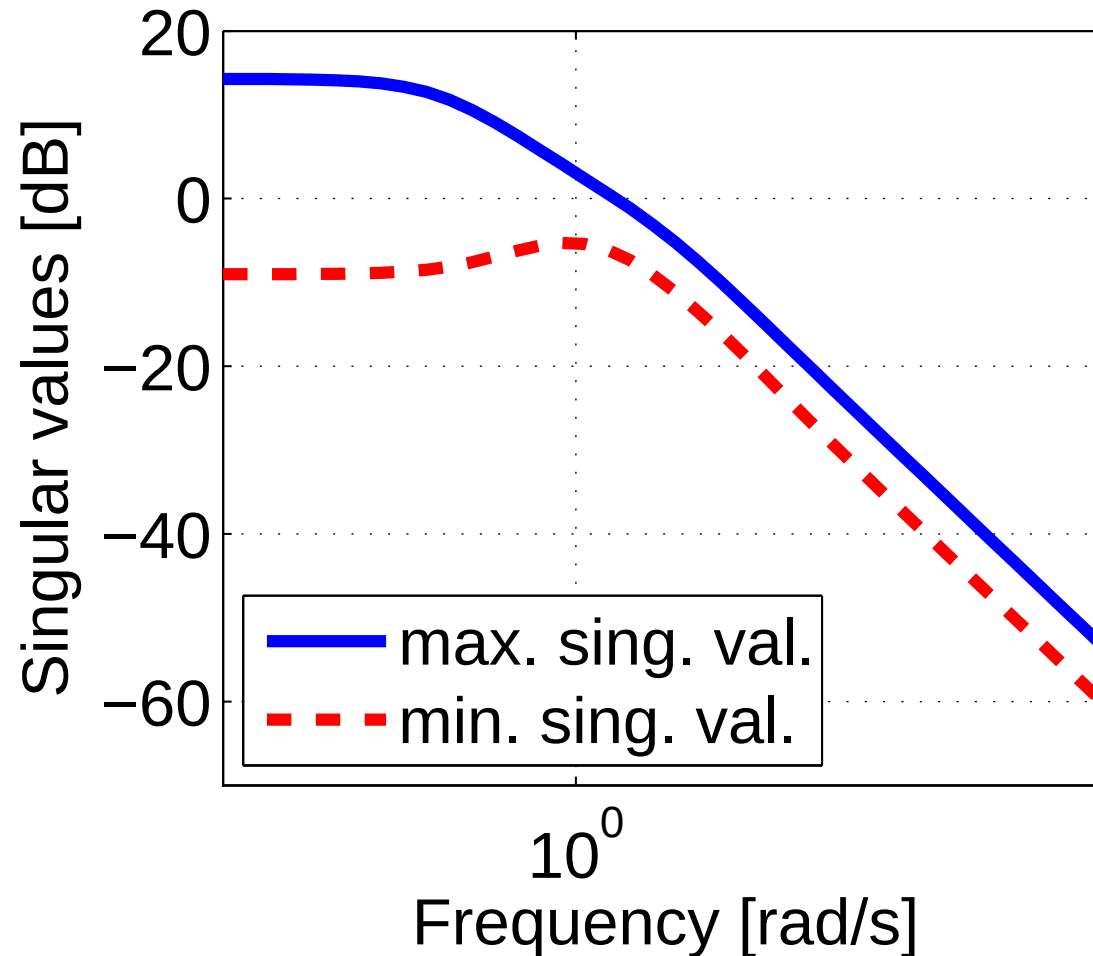
but simulation of real closed loop  
shows instability!



# Singular Value Diagrams

Example:  $G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{0.5}{s+3} \\ \frac{10}{s^2+10s+2} & \frac{-2}{s+2} \end{bmatrix}$

Singular value plot



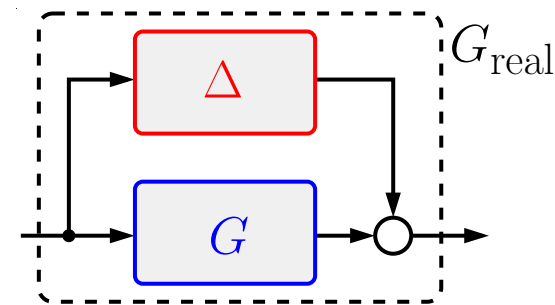
# Typical Unstructured Uncertainties

- **Additive uncertainty:**

$$G_{\text{real}}(s) = G(s) + \Delta(s)$$

Useful for:

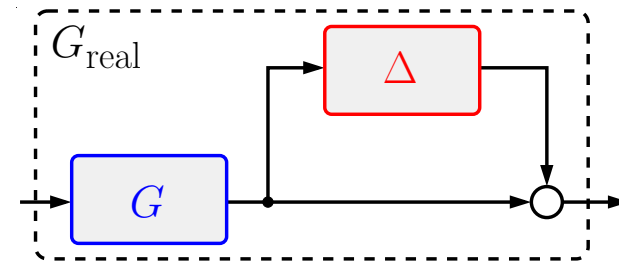
high-freq. (HF) plant dynamics, uncertain zeros



- **Multiplicative output uncertainty:**

$$G_{\text{real}}(s) = (I + \Delta(s))G(s)$$

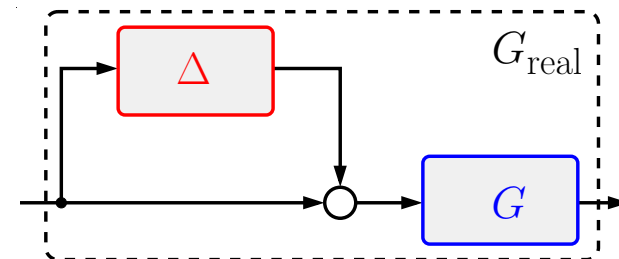
Useful for: HF plant dynamics, sensor errors, uncertain zeros



- **Multiplicative input uncertainty:**

$$G_{\text{real}}(s) = G(s)(I + \Delta(s))$$

Useful for: HF plant dynamics, actuator errors, uncertain zeros



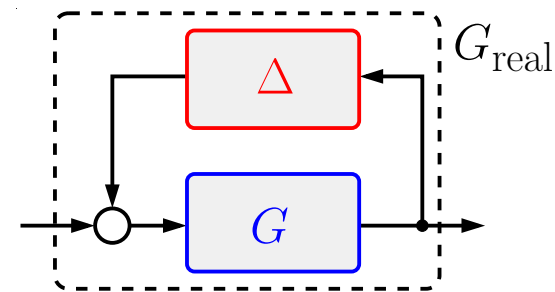
# Typical Unstructured Uncertainties

- **Inverse additive uncertainty:**

$$G_{\text{real}}(s) = G(s)(I - \Delta(s)G(s))^{-1}$$

Useful for:

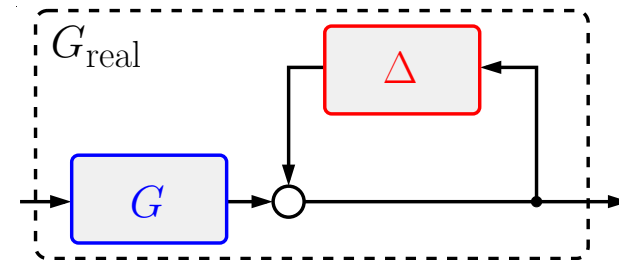
low-freq. (LF) parameter errors, uncertain poles



- **Inverse mult. output uncertainty:**

$$G_{\text{real}}(s) = (I - \Delta(s))^{-1}G(s)$$

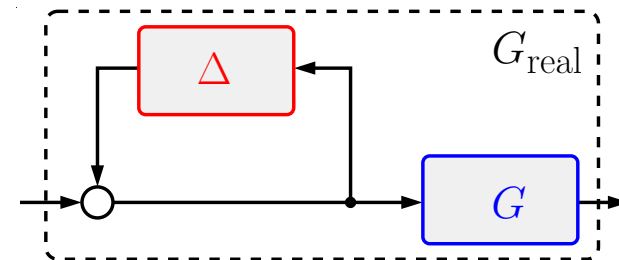
Useful for: dito



- **Inverse mult. input uncertainty:**

$$G_{\text{real}}(s) = G(s)(I - \Delta(s))^{-1}$$

Useful for: dito



# Normalization of $\Delta$

What do we do if we have  $\bar{\sigma}(\Delta(j\omega)) < l(\omega) \forall \omega$  for some frequency-dependent function  $l : \mathbb{R} \rightarrow \mathbb{R}^+$  instead of  $\bar{\sigma}(\Delta(j\omega)) < 1 \forall \omega$ ?

Normalize the uncertainty  $\Delta$ :

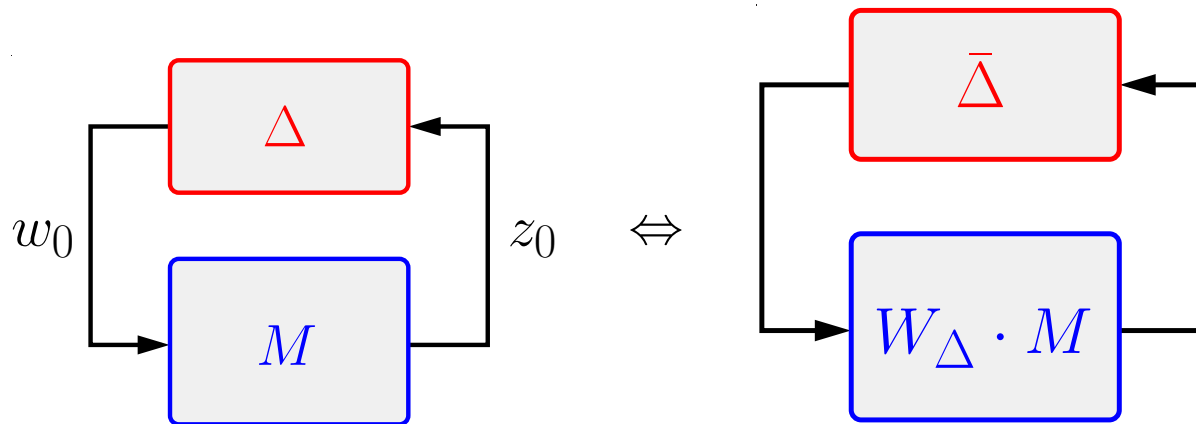
1. Suppose  $\dim(\Delta) = q_0 \times p_0$
2. Find a stable, proper, minimum-phase SISO system  $w_\Delta(s)$   
with  $\bar{\sigma}(w_\Delta(j\omega)) = |w_\Delta(j\omega)| = l(\omega)$   
(or  $\bar{\sigma}(w_\Delta(j\omega)) = |w_\Delta(j\omega)| \geq l(\omega)$ )
3. Set  $W_\Delta(s) = w_\Delta(s) \cdot I$ , where size of  $I$  is  $p_0 \times p_0$
4. Suppose we have another uncertainty  $\bar{\Delta}$  with  $\bar{\sigma}(\bar{\Delta}(j\omega)) < 1 \forall \omega$

# Normalization of $\Delta$

5. Substitute  $\Delta(s) := \bar{\Delta}(s)W_{\Delta}(s)$

Hence it holds that  $\underbrace{\bar{\sigma}(\bar{\Delta}(j\omega)W_{\Delta}(j\omega))}_{\Delta(j\omega)} \leq \bar{\sigma}(\bar{\Delta}(j\omega)) \cdot \underbrace{\bar{\sigma}(W_{\Delta}(j\omega))}_{l(\omega)} < l(\omega)$

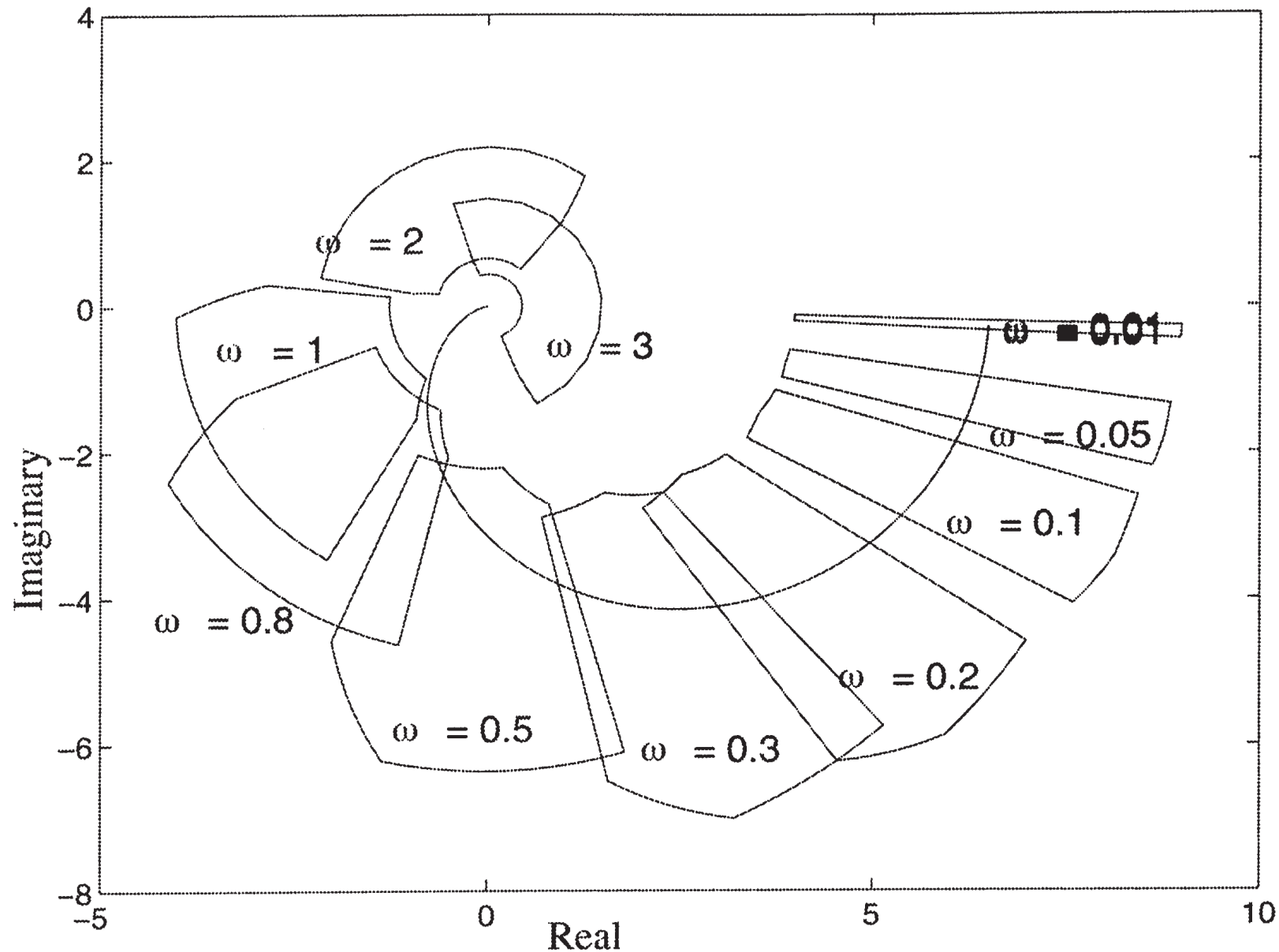
6. Rearrange the  $M$ - $\Delta$ -structure:



7. Use  $(W_{\Delta}M, \bar{\Delta})$  instead of  $(M, \Delta)$ .

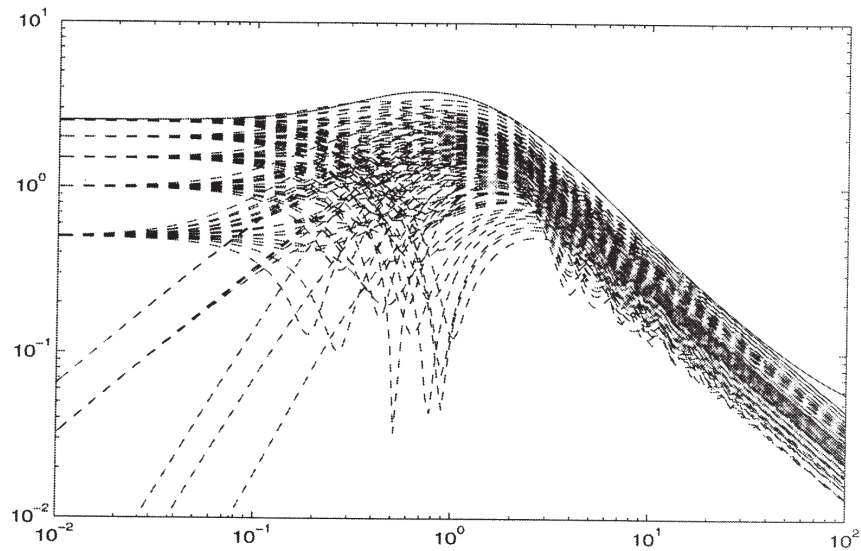
# Unsicherheitsbeschreibung: Nyquist-Diag.

Beispiel 8.2 aus K.Zhou, Essentials of robust control.

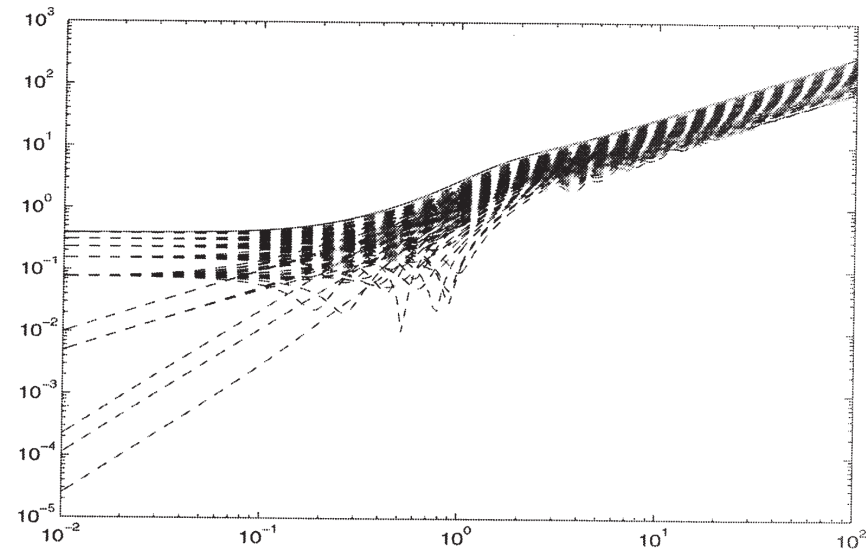


# Unsicherheitsbeschreibung

## Additive Unsicherheit



## Multiplikative Unsicherheit



# A Robust Stability Theorem

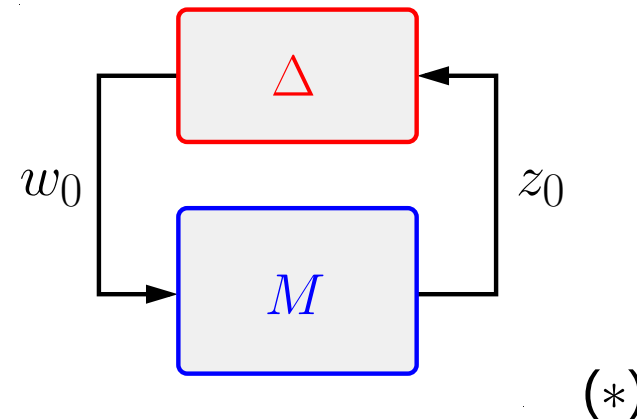
**Def.:** A property is called *robust* against a class of uncertainties / model errors, if the property holds in face of these uncertainties.

## Theorem:

Suppose that  $M$  is a stable and proper LTI system, and consider the class of stable and proper uncertainties  $\Delta$  with  $\|\Delta\|_\infty < 1$ .

Then the following statements are equivalent:

- (i) The closed-loop interconnection (\*) is robustly stable.
- (ii) The closed-loop interconnection (\*) is internally stable for all considered uncertainties  $\Delta$ .
- (iii)  $\bar{\sigma}(M(j\omega)) \leq 1 \quad \forall \omega$
- (iv)  $\|M\|_\infty \leq 1$



# A Robust Stability Theorem

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## Remarks:

- $\Delta$  is any dynamical system satisfying the assumed norm bound.  
Actually,  $\Delta$  can be a linear system with an arbitrary number of states, it can be just a complex matrix  $\in \mathbb{C}^{q_0 \times p_0}$ , or it can be a time-varying or nonlinear system (with corresponding stability notions).  
Hence, usually a much larger class of uncertainties than intended is taken into account.
- The theorem is necessary and sufficient.  
Sufficiency means:  $\|M\|_\infty \leq 1 \Rightarrow$  robust stability.  
Necessity means:  $\|M\|_\infty > 1 \Rightarrow$  there exists at least one uncertainty in the considered class that results in instability. It does not imply that the closed loop with  $G_{real}$  is necessarily unstable.